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LETTER TO THE EDITOR

The singularity of the background scalar field and the fractional fermion charge†

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Abstract. It is shown that the singularity of the background scalar field, even if this field is topological trivial and/or localised in a finite region, will give rise to fractional fermionic charge of the vacuum sector of a quantised Dirac field in $(1+1)$ dimensions. The physical effect of the singularity of the background field is discussed.

Since the pioneering work of Jackiw and Rebbi [1], much effort has been devoted to studying the charge fractionalisation of a quantised Dirac field interacting with a background field in $(1+1)$ and $(1+3)$ dimensions. They include the adiabatic approximation by Goldstone and Wilczek [2], the anomalous commutors by Bardeen *et al* [3], the Jost function by Blankenbecler and Boyanovsky [4], the Levinson theorem by Ma *et al* [5], and others (see, e.g., [6]). At zero temperature, they come to the same conclusions that: (i) the zero modes of a fermion interacting with a scalar field in $(1+1)$ dimensions and with a monopole field in $(1+3)$ dimensions give rise to states with a fermion number $\frac{1}{2}$; (ii) if the background field is topologically non-trivial, the Dirac Hamiltonian does possess normalisable zero-energy states and fractional charge will occur. Recently, many authors have tried to extend this discussion to finite temperature, using different methods, for example, the trace identity method by Niemi and Semonoff [7], chemical potential by Soni and Baskaran [8] and the generalised Bogoliubov transformation by two of the present authors [9].

In order to make the conclusions at zero temperature transparently, Jackiw and his co-workers [10] analysed the limit of infinite soliton–antisoliton separation $L\to\infty$ carefully. They pointed out that the total charge in a soliton–antisoliton system has integer eigenvalues, but in the limit $L\to\infty$, the localised charge operator has fractional eigenvalues, without fluctuations. Furthermore, using the Green function method, Bernstein and Brown [11] examined the $(1+1)$-dimensional fermionic theory quantised in a finite region and found that the vacuum charge *cannot* be an irrational number for a *finite* region, but when the infinite volume limit is taken the boundary charge escapes to infinity, leaving behind a fractional charge.

The present letter evolves from an attempt to investigate the behaviour of a fermion field with scalar coupling in a finite region. We find that if the background scalar field has a *singularity* in this finite region, even though the scalar field is localised in a finite volume, the vacuum charge can be a fractional charge, without fluctuation.

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In order to justify this conclusion, let us consider a simple model. The (1+1)-dimensional Dirac equation with the prescription $\alpha=\sigma_2$, $\beta=\sigma_1$ is

$$E\psi=[\alpha p+\beta\phi(x)]\psi \tag{1}$$

where $\phi(x)$ is a scalar background field chosen as

$$\phi(x)=\mu(\theta(-a-x)-\mu\theta(x+a)\theta(-x-\varepsilon)$$
$$+\mu\theta(x+\varepsilon)\theta(-x+\varepsilon)-\mu\theta(x-\varepsilon)\theta(-x+a)+\mu\theta(x-a) \tag{2}$$

(see figure 1) where $\theta(x)$ is the step function

$$\theta(x)=\begin{cases}1 & x>0\\ 0 & x<0.\end{cases}$$

We solve the Dirac equation (1) with this potential $\phi(x)$ first and then make the potential barrier approach infinity ($\mu\to\infty$) and the barrier length ε approach zero ($\varepsilon\to0$). In this limit, the scalar potential $\phi(x)$ is localised in a region $|x|<a$ and it has a singularity at $x=0$.

We decompose the eigenfunction $\psi(x)$ into two components $\psi=\binom{u}{v}$ and find

$$\begin{aligned}Eu&=-v'+\phi v\\ Ev&=u'+\phi u\end{aligned} \tag{3}$$

or

$$\begin{aligned}E^2u&=-u''+(\phi^2-\phi')u\\ E^2v&=-v''+(\phi^2+\phi')v.\end{aligned} \tag{4}$$

Noting that

$$\phi^2\pm\phi'=\mu^2\mp2\mu\delta(x+a)\pm2\mu\delta(x+\varepsilon)\mp2\mu\delta(x-\varepsilon)\pm2\mu\delta(x-a) \tag{5}$$

we get the boundary conditions of v

$$\begin{array}{lll} x=-a & v_{-a^+}=v_{-a^-} & v'_{-a^+}-v'_{-a^-}=-2\mu v_{-a}\\ x=-\varepsilon & v_{-\varepsilon^+}=v_{-\varepsilon^-} & v'_{-\varepsilon^+}-v'_{-\varepsilon^-}=2\mu v_{-\varepsilon}\\ x=\varepsilon & v_{\varepsilon^+}=v_{\varepsilon^-} & v'_{\varepsilon^+}-v'_{\varepsilon^-}=-2\mu v_{+\varepsilon}\\ x=a & v_{a^+}=v_{a^-} & v'_{a^+}-v'_{a^-}=2\mu v_a. \end{array} \tag{6}$$

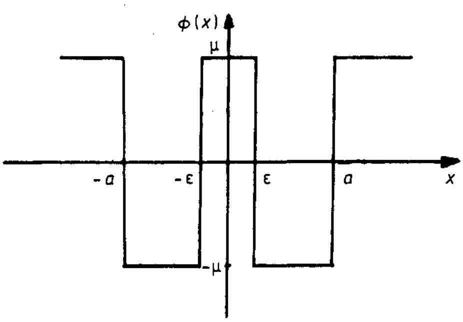

Figure 1. The background field $\phi(x)$.

The solutions of v which satisfy the boundary conditions (6) are

$$
v=\begin{cases}
A\,\mathrm{e}^{kx} & x<-a \\
A\left[\left(1-\dfrac{\mu}{k}\right)\mathrm{e}^{kx}+\dfrac{\mu}{k}\,\mathrm{e}^{-2ka}\,\mathrm{e}^{-kx}\right] & -a\leqslant x<-\varepsilon \\
A\left[\left(1-\dfrac{\mu^2}{k^2}+\dfrac{\mu^2}{k^2}\,\mathrm{e}^{-2ka+2k\varepsilon}\right)\mathrm{e}^{kx}+\dfrac{\mu}{k}\left(1-\dfrac{\mu}{k}\right)(\mathrm{e}^{-2ka}-\mathrm{e}^{-2k\varepsilon})\,\mathrm{e}^{-kx}\right] & -\varepsilon\leqslant x<\varepsilon \\
A\left\{\left(1-\dfrac{\mu}{k}\right)\left[\left(1-\dfrac{\mu^2}{k^2}+\dfrac{\mu^2}{k^2}\,\mathrm{e}^{-2ka+2k\varepsilon}\right)+\dfrac{\mu^2}{k^2}(\mathrm{e}^{-2k\varepsilon}-\mathrm{e}^{-2ka})\,\mathrm{e}^{-2k\varepsilon}\right]\mathrm{e}^{kx}\right. & \\
\quad\left.+\dfrac{\mu}{k}\left[\left(1-\dfrac{\mu^2}{k^2}+\dfrac{\mu^2}{k^2}\,\mathrm{e}^{-2ka+2k\varepsilon}\right)\mathrm{e}^{2k\varepsilon}+\left(1-\dfrac{\mu^2}{k^2}\right)(\mathrm{e}^{-2ka}-\mathrm{e}^{-2k\varepsilon})\right]\mathrm{e}^{-kx}\right\} & \varepsilon\leqslant x<a \\
A\left\{\left(1-\dfrac{\mu^2}{k^2}\right)\left[\left(1-\dfrac{\mu}{k}\right)\mathrm{e}^{2ka}+\dfrac{\mu}{k}\,\mathrm{e}^{-2ka}\right]+\dfrac{\mu^2}{k^2}\left[\left(1-\dfrac{\mu}{k}\right)\mathrm{e}^{2ka-4k\varepsilon}+\dfrac{\mu}{k}\,\mathrm{e}^{-ka+4k\varepsilon}\right]\right. & \\
\quad\left.+\dfrac{\mu}{k}\left(1-\dfrac{\mu}{k}\right)\left(1+\dfrac{2\mu}{k}\right)(\mathrm{e}^{2k\varepsilon}-\mathrm{e}^{-2k\varepsilon})\right\}\mathrm{e}^{-kx} & a\leqslant x
\end{cases}
\tag{7}
$$

where

$$
k^2=\mu^2-E^2 \tag{8}
$$

and A is a normalised constant. Another component u can be obtained from equation (3):

$$
u=\begin{cases}
A\left(\dfrac{\mu-k}{\mu+k}\right)^{1/2}\mathrm{e}^{kx} & x<-a \\
-A\left[\left(1-\dfrac{\mu}{k}\right)\left(\dfrac{\mu+k}{\mu-k}\right)^{1/2}\mathrm{e}^{kx}+\dfrac{\mu}{k}\left(\dfrac{\mu-k}{\mu+k}\right)^{1/2}\mathrm{e}^{-2ka}\,\mathrm{e}^{-kx}\right] & -a\leqslant x<-\varepsilon \\
A\left[\left(1-\dfrac{\mu^2}{k^2}+\dfrac{\mu^2}{k^2}\,\mathrm{e}^{-2ka+2k\varepsilon}\right)\left(\dfrac{\mu-k}{\mu+k}\right)^{1/2}\mathrm{e}^{kx}+\dfrac{\mu}{k}\left(1-\dfrac{\mu}{k}\right)(\mathrm{e}^{-2ka}-\mathrm{e}^{-2k\varepsilon})\right. & \\
\qquad\left.\times\left(\dfrac{\mu+k}{\mu-k}\right)^{1/2}\mathrm{e}^{-kx}\right] & -\varepsilon\leqslant x<\varepsilon \\
-A\left\{\left(1-\dfrac{\mu}{k}\right)\left[\left(1-\dfrac{\mu^2}{k^2}+\dfrac{\mu^2}{k^2}\,\mathrm{e}^{-2ka+2k\varepsilon}\right)+\dfrac{\mu^2}{k^2}(\mathrm{e}^{-2k\varepsilon}-\mathrm{e}^{-2ka})\,\mathrm{e}^{-2k\varepsilon}\right]\left(\dfrac{\mu+k}{\mu-k}\right)^{1/2}\mathrm{e}^{kx}\right. & \\
\quad\left.+\dfrac{\mu}{k}\left[\left(1-\dfrac{\mu^2}{k^2}+\dfrac{\mu^2}{k^2}\,\mathrm{e}^{-2ka+2k\varepsilon}\right)\mathrm{e}^{2k\varepsilon}+\left(1-\dfrac{\mu^2}{k^2}\right)(\mathrm{e}^{-2ka}-\mathrm{e}^{-2k\varepsilon})\right]\right\} & \\
\qquad\times\left(\dfrac{\mu-k}{\mu+k}\right)^{1/2}\mathrm{e}^{-kx} & \varepsilon\leqslant x<a \\
A\left\{\left(1-\dfrac{\mu^2}{k^2}\right)\left[\left(1-\dfrac{\mu}{k}\right)\mathrm{e}^{2ka}+\dfrac{\mu}{k}\,\mathrm{e}^{-2ka}\right]+\dfrac{\mu^2}{k^2}\left[\left(1-\dfrac{\mu}{k}\right)\mathrm{e}^{2ka-4k\varepsilon}+\dfrac{\mu}{k}\,\mathrm{e}^{-2ka+4k\varepsilon}\right]\right. & \\
\quad\left.+\dfrac{\mu}{k}\left(1-\dfrac{\mu}{k}\right)\left(1+\dfrac{2\mu}{k}\right)(\mathrm{e}^{2k\varepsilon}-\mathrm{e}^{-2k\varepsilon})\right\}\left(\dfrac{\mu+k}{\mu-k}\right)^{1/2}\mathrm{e}^{-kx} & a\leqslant x.
\end{cases}
\tag{9}
$$

The equation of the energy spectra is

$$\left(1-\frac{\mu^2}{k^2}\right)\left[\left(1-\frac{\mu^2}{k^2}\right)\mathrm{e}^{2ka}-\frac{\mu^2}{k^2}\mathrm{e}^{-2ka}\right]+\frac{\mu^2}{k^2}\left[\left(1-\frac{\mu^2}{k^2}\right)\mathrm{e}^{2ka-4k\varepsilon}+\frac{\mu^2}{k^2}\mathrm{e}^{-2ka+4k\varepsilon}\right]$$
$$+2\frac{\mu^2}{k^2}\left(1-\frac{\mu^2}{k^2}\right)(\mathrm{e}^{2k\varepsilon}-\mathrm{e}^{-2k\varepsilon})=0. \tag{10}$$

It can easily be shown that solutions (7) and (9) cannot give us a fractional fermion charge, without fluctuations.

Now we are in a position to consider the limit of $\mu\to\infty$ and $\varepsilon\to 0$. In this condition, equation (10) becomes

$$\mu^2-k^2=E^2=\mu^2\,\mathrm{e}^{-4ka} \tag{11}$$

Obviously, when $\mu\to\infty$, i.e. $k\to\infty$, equation (11) has zero modes.

Using the method given by Jackiw *et al* (10), we expand the second quantised ψ operator as

$$\psi=\sum_n\left[a_n\psi_n(x)\exp(-\mathrm{i}E_nt)+b_n^+\sigma_3\psi_n^*(x)\exp(\mathrm{i}E_nt)\right] \tag{12}$$

where

$$[a_n, a_{n'}^+]_+=[b_n, b_{n'}^+]_+=\delta_{nn'}. \tag{13}$$

In the limit of $\mu\to\infty$ and $\varepsilon\to 0$, we can prove

$$\begin{aligned}\lim_{\mu\to\infty,\varepsilon\to 0}\int_x f(u_n^2+v_n^2)\,\mathrm{d}x&=\tfrac{1}{2}\\ \lim_{\mu\to\infty,\varepsilon\to 0}\int_x f(u_n^2-v_n^2)\,\mathrm{d}x&=-\tfrac{1}{2}\end{aligned} \tag{14}$$

where f is the sampling function which we choose as

$$f=\theta(-x)\theta(x+a). \tag{15}$$

Performing the Bogoliubov transformation, we get

$$N_{\mathrm{L}}=\int_{-\infty}^{+\infty}\theta(-x)\theta(x+a){:}\psi^+\psi{:}\,\mathrm{d}x=\sum_n\tfrac{1}{2}(a_n^+a_n+b_n^+b_n). \tag{16}$$

The fluctuation of N can be shown from equation (14) as

$$(\Delta N_{\mathrm{f}})^2=\int_x f^2(u_B^2+v_B^2)\,\mathrm{d}x-\left(\int_x f(u_B^2+v_B^2)\right)^2\mathrm{d}x-\left(\int_x f(u_B^2-v_B^2)\right)^2\mathrm{d}x$$
$$+\iint_{kk'}\left|\int_x f(u_ku_{k'}-v_kv_{k'})\right|^2\mathrm{d}x\,\mathrm{d}k\,\mathrm{d}k'$$
$$=\tfrac{1}{2}-(\tfrac{1}{2})^2-(-\tfrac{1}{2})^2+0=0. \tag{17}$$

This means that we obtain a fractional fermion charge without fluctuations provided the background field has singularity.

The physical picture of our conclusion is obvious: as discussed by Jackiw *et al* the quantum fractionisation in soliton system is of course to be contrasted with the situation of a two-centred molecule with one electron passing between the two atoms. Although the expectation value of the charge near one atom of the latter is $\frac{1}{2}$, we can prove that the fluctuation $(\Delta N)^2$ is non-zero since the ground states are degenerate. Then we cannot get a fractional fermion number. However, if we have an infinite barrier between these two atoms and if the electron cannot penetrate this barrier, the wavefunctions of the right-hand side and the left-hand side will be independent, and the degeneracy of the exchange symmetry between the left and the right will be cancelled. The singularity of the scalar field in the Dirac equation plays the role of an infinite barrier, since the Klein paradox will not happen for a scalar potential [12].

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